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LETTER TO THE EDITOR

Sign-time distribution for a random walker with a drifting boundary

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Abstract

We present a derivation of the exact sign-time distribution for a random walker in the presence of a boundary moving with constant velocity.

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There has been tremendous interest within the statistical physics community over the past five years in the persistence properties of various simple systems [1]. Aside from the persistence probability itself, another quantity of great interest is the 'sign-time' distribution (STD) [2, 3] (also known as the 'occupation-time' or 'residence-time' distribution). It seems that STDs are very difficult to calculate, even for the simplest systems. Famous examples for which the STD is known exactly are the pure random walk, yielding the arcsine law of Lévy [4, 5], and the renewal process, originally studied by Lamperti [6], and recently revisited [7, 8]. There have been concerted efforts recently to find the STD for a generalized random walk model [9, 10], which have met with some success, although an exact solution is still not known. Here we consider the STD for a random walker in the presence of a boundary moving with constant velocity (which is equivalent to a random walker with a constant drift velocity in the presence of a stationary boundary). The first passage probability [11] and the persistence probability [12] for this problem are both known. However, to the author's knowledge, the STD has not been previously calculated. The purpose of this Letter is to present a relatively straightforward derivation and analysis of this latter quantity.

We consider a random walker whose position is denoted by $x(t)$, which satisfies the equation of motion

$$\dot{x} = \xi(t) \tag{1}$$

with $x(0) = 0$, and $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = D\delta(t-t')$. In addition, we consider a moving boundary whose position is denoted by $b(t)$. In the main part of this Letter we shall consider linear motion of the boundary, i.e. $b(t) = vt$.

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The sign-time $\tau_\xi(t)$ is defined as the proportion of elapsed time t the walker has spent on the positive side of the boundary:

$$\tau_\xi(t) = \int_0^t \mathrm{d}t'\, \Theta(x(t') - b(t')) \tag{2}$$

where $\Theta(x)$ is the step function. We are interested in calculating the STD $P(\tau, t)$ defined by

$$P(\tau, t) = \left\langle \delta(\tau - \tau_\xi(t)) \right\rangle. \tag{3}$$

By making an integral representation of $P(\tau, t)$, we can reduce the problem to a calculation of the moments of the STD:

$$P(\tau, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega\tau} \left\langle \exp\left[\mathrm{i}\omega\tau_\xi(t)\right]\right\rangle \tag{4}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega\tau} \sum_{n=0}^{\infty} \frac{(\mathrm{i}\omega)^n}{n!} \left\langle \tau_\xi(t)^n \right\rangle. \tag{5}$$

To this end we introduce a convenient double-integral representation of the step function

$$\Theta(x) = \int_0^\infty \mathrm{d}u\, \delta(x - u) = \int_0^\infty \mathrm{d}u \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}k(x-u)}. \tag{6}$$

Utilizing this in the definition of the sign-time, we can write the nth moment as the following time-ordered expression:

$$\left\langle \tau_\xi(t)^n \right\rangle = n! \int \mathcal{D}t(n)\, \langle \Theta(x(t_1) - b(t_1)) \cdots \Theta(x(t_n) - b(t_n)) \rangle \tag{7}$$

$$= n! \int \mathcal{D}t(n) \int \mathcal{D}u(n) \int \mathcal{D}k(n)\, \mathrm{e}^{-\mathrm{i}k_1(u_1 + b(t_1))} \cdots \mathrm{e}^{-\mathrm{i}k_n(u_n + b(t_n))} \left\langle \mathrm{e}^{\mathrm{i}k_1 x(t_1)} \cdots \mathrm{e}^{\mathrm{i}k_n x(t_n)} \right\rangle \tag{8}$$

where the multiple integral symbols are defined via

$$\int \mathcal{D}t(n) \equiv \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \tag{9}$$

$$\int \mathcal{D}u(n) \equiv \int_0^\infty \mathrm{d}u_1 \cdots \int_0^\infty \mathrm{d}u_n \tag{10}$$

$$\int \mathcal{D}k(n) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{\mathrm{d}k_n}{2\pi}. \tag{11}$$

Writing the solution of equation (1) as $x(t) = \int_0^t \mathrm{d}t'\, \xi(t')$, the average on the right-hand side of equation (8) is easily performed to give

$$\left\langle \mathrm{e}^{\mathrm{i}k_1 x(t_1)} \cdots \mathrm{e}^{\mathrm{i}k_n x(t_n)} \right\rangle = \exp\left\{-\frac{D}{2}\left[k_1^2(t_1 - t_2) + (k_1 + k_2)^2(t_2 - t_3) + \cdots + (k_1 + \cdots + k_n)^2 t_n\right]\right\}. \tag{12}$$

Inserting this expression into equation (8) and changing variables via ($p_1 = k_1, \ldots, p_n = k_1 + \cdots + k_n$), we find the integrals over $\{p_i\}$ decompose and may be performed, giving us

$$\left\langle \tau_\xi(t)^n \right\rangle = n! \int \mathcal{D}t(n) \int \mathcal{D}u(n)\, G(u_1 + b(t_1) - u_2 - b(t_2), t_1 - t_2) \cdots G(u_n + b(t_n), t_n) \quad (13)$$

where $G(u, t)$ is the random walk propagator

$$G(u, t) = (2\pi Dt)^{-1/2} \exp[-u^2/2Dt]. \quad (14)$$

We shall now focus on the case of a boundary drifting with constant velocity $v \geqslant 0$, i.e. we write $b(t) = vt$. (The case of $v < 0$ may be recovered by taking $v \to -v$ and $\tau \to t - \tau$.) We note the appearance of a fundamental rate constant $\gamma \equiv v^2/2D$. It is convenient to introduce the function $H(u, t) = G(u + vt, t)$. By considering the Laplace transform of the nth moment, and invoking the Laplace transform convolution theorem iteratively, we find

$$\mathcal{L}_{s|t}\left[\left\langle \tau_\xi(t)^n \right\rangle\right] = \frac{n!}{s} \int \mathcal{D}u(n)\, \hat{H}(u_1 - u_2, s) \cdots \hat{H}(u_{n-1} - u_n, s)\hat{H}(u_n, s) \quad (15)$$

where the Laplace transform operator is denoted by

$$\hat{f}(s) = \mathcal{L}_{s|t}[f(t)] = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} f(t). \quad (16)$$

Now, $H(u, t) = \mathrm{e}^{-uv/D - \gamma t} G(u, t)$ and consequently $\hat{H}(u, s) = e^{-uv/D}\hat{G}(u, s + \gamma)$. The Laplace transform of $G(u, t)$ is given by

$$\hat{G}(u, s) = (2sD)^{-1/2} \exp[-(2s/D)^{1/2}|u|]. \quad (17)$$

Substituting the explicit form for $\hat{H}(u, s)$ into equation (15) and rescaling the integration variables we find

$$\mathcal{L}_{s|t}\left[\left\langle \tau_\xi(t)^n \right\rangle\right] = \frac{n!}{s[2(s+\gamma)]^n} \int \mathcal{D}u(n)\, \mathrm{e}^{-\alpha(s)u_1} \exp\left(-|u_1 - u_2| - \cdots - |u_{n-1} - u_n| - |u_n|\right) \quad (18)$$

where we have defined $\alpha(s) = (\gamma/(s + \gamma))^{1/2}$. Equation (18) can be conveniently rewritten, for $n \geqslant 1$, as

$$\mathcal{L}_{s|t}\left[\left\langle \tau_\xi(t)^n \right\rangle\right] = \frac{n!}{s[2(s+\gamma)]^n} \int_0^\infty \mathrm{d}u_1\, \mathrm{e}^{-\alpha(s)u_1} J_{n-1}(u_1) \quad (19)$$

where the function $J_n(u)$ satisfies the integro-difference equation

$$J_n(u) = \int_0^\infty \mathrm{d}u'\, \mathrm{e}^{-|u-u'|} J_{n-1}(u') \quad (20)$$

with $J_0 = \mathrm{e}^{-u}$. This equation for $J_n(u)$ can be solved using generating function methods, and the details are relegated to the appendix. The end result is

$$J_n(u) = \oint_C \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left(\frac{1 - (1 - 2z)^{1/2}}{z^{n+2}}\right) \exp[-(1 - 2z)^{1/2} u] \quad (21)$$

where the contour C encircles the origin.

We now substitute this solution for $J_n(u)$ into equation (19) and perform the integral over u_1. Using the fact that $\oint_C \mathrm{d}z/z^{n+1} = 0$ for $n \geqslant 1$ allows us to rewrite the integrand so as to obtain

$$\mathcal{L}_{s|t}\left[\langle\tau_\xi(t)^n\rangle\right] = \frac{n!(1+\alpha(s))}{s[2(s+\gamma)]^n}\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\frac{1}{z^{n+1}[\alpha(s)+(1-2z)^{1/2}]}. \tag{22}$$

The integral around the contour C may be re-expressed as an integral across the cut emanating from the branch point at $z = 1/2$ with the result

$$\mathcal{L}_{s|t}\left[\langle\tau_\xi(t)^n\rangle\right] = \frac{n!(1+\alpha(s))}{\pi s(s+\gamma)^n}\int_0^\infty \mathrm{d}q\,\frac{q^{1/2}}{(1+q)^{n+1}(\alpha(s)^2+q)}. \tag{23}$$

It can be explicitly checked that this expression is also correct for $n = 0$, namely, that it gives the result $1/s$.

It is possible to invert the Laplace transform at this point, but it is more convenient to postpone this operation and instead introduce the integral representation for $n!$ via

$$n! = \int_0^\infty \mathrm{d}y\, y^n \mathrm{e}^{-y}. \tag{24}$$

Then equation (23) takes the form

$$\mathcal{L}_{s|t}\left[\langle\tau_\xi(t)^n\rangle\right] = \frac{(1+\alpha(s))}{\pi s}\int_0^\infty \mathrm{d}q\,\frac{q^{1/2}}{(1+q)(\alpha(s)^2+q)}\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-y}\left[\frac{y}{(s+\gamma)(1+q)}\right]^n. \tag{25}$$

Referring to equation (5) we have

$$\mathcal{L}_{s|t}\left[P(\tau,t)\right] = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega\tau}\sum_{n=0}^{\infty}\frac{(\mathrm{i}\omega)^n}{n!}\mathcal{L}_{s|t}\left[\langle\tau_\xi(t)^n\rangle\right]. \tag{26}$$

Substituting equation (25) into equation (26), we can explicitly perform the sum over n which allows us to complete the integral over ω yielding a Dirac δ-function with argument $(\tau - y/(s+\gamma)(1+q))$, which in turn allows us to trivially perform the integral over y. This leaves us with (after rescaling the q integration variable)

$$\mathcal{L}_{s|t}\left[P(\tau,t)\right] = \frac{\alpha(s)(1+\alpha(s))(s+\gamma)}{\pi s}\mathrm{e}^{-(s+\gamma)\tau}\int_0^\infty \mathrm{d}q\,\frac{q^{1/2}\mathrm{e}^{-\gamma\tau q}}{(1+q)}. \tag{27}$$

It is worth noting the clean disappearance of the Laplace transform variable s from inside the q-integral, which allows us to complete the calculation easily from this point. Formally inverting the Laplace transform gives us

$$P(\tau,t) = N(\gamma\tau)\mathcal{L}^{-1}_{t|s}\left[\frac{\alpha(s)(1+\alpha(s))(s+\gamma)}{s}\mathrm{e}^{-s\tau}\right] \tag{28}$$

where we have defined

$$N(\beta) \equiv \frac{\mathrm{e}^{-\beta}}{\pi}\int_0^\infty \mathrm{d}q\,\frac{q^{1/2}\mathrm{e}^{-\beta q}}{(1+q)} = \frac{\mathrm{e}^{-\beta}}{(\pi\beta)^{1/2}} - \mathrm{erfc}(\beta^{1/2}) \tag{29}$$

with $\mathrm{erfc}(z)$ the complementary error function [13]. On performing the inverse Laplace transform in equation (28) we obtain

$$P(\tau,t) = \gamma N(\gamma\tau)[2 + N(\gamma(t-\tau))]. \tag{30}$$

This may be cast into a more symmetrical form by defining the function F^+ and its companion F^- via

$$F^{\pm}(\beta) = \mathrm{e}^{-\beta} \mp (\pi\beta)^{1/2}\mathrm{erfc}(\pm\beta^{1/2}). \tag{31}$$

We then have our final result in the form

$$P(\tau, t) = \frac{F^+(\gamma\tau)F^-(\gamma(t-\tau))}{\pi(\tau(t-\tau))^{1/2}}. \tag{32}$$

It is interesting to note that the distribution is a product of a function of τ and a function of $(t-\tau)$. It is often convenient to express the STD in dimensionless variables. Defining $\phi = \tau/t \in (0, 1)$ and $\eta = \gamma t$ we have $P(\tau, t)\mathrm{d}\tau = \Psi(\phi, \eta)\mathrm{d}\phi$ and consequently

$$\Psi(\phi, \eta) = \frac{F^+(\eta\phi)F^-(\eta(1-\phi))}{\pi(\phi(1-\phi))^{1/2}}. \tag{33}$$

We briefly examine some limits of the STD. For small velocity, or small times, we have $\eta \ll 1$, and thus

$$\Psi(\phi, \eta) = \frac{1}{\pi[\phi(1-\phi)]^{1/2}} - \left(\frac{\eta}{\pi}\right)^{1/2}\left(\frac{1}{(1-\phi)^{1/2}} - \frac{1}{\phi^{1/2}}\right) + \eta\left[\frac{1}{\pi(\phi(1-\phi))^{1/2}} - 1\right] + O(\eta^{3/2}) \tag{34}$$

where the leading term is the arcsine law of Lévy, as required.

The limiting forms of Ψ for $\phi \to 0$ and $\phi \to 1$ are given below, along with the various forms of large-η behaviour:

$$\Psi(\phi, \eta) \sim \begin{cases} \dfrac{F^-(\eta)}{\pi\phi^{1/2}} & \phi \to 0 \\ \dfrac{F^+(\eta)}{\pi(1-\phi)^{1/2}} & \phi \to 1 \\ \dfrac{\mathrm{e}^{-\eta\phi}}{(\pi\eta\phi^3)^{1/2}} & \eta \to \infty,\ \eta\phi \to \infty,\ \eta(1-\phi) \to \infty \\ 2\left(\dfrac{\eta}{\pi\phi}\right)^{1/2} & \eta \to \infty,\ \phi \to 0,\ \eta\phi \to 0 \\ \dfrac{\mathrm{e}^{-\eta}}{2\pi\eta(1-\phi)^{1/2}} & \eta \to \infty,\ \phi \to 1,\ \eta(1-\phi) \to 0\,. \end{cases} \tag{35}$$

The distribution has a single minimum whose position $\phi_{\min}(\eta)$ approaches unity as $\phi_{\min}(\eta) \sim 1 - r/\eta$ for $\eta \gg 1$, where the universal number r satisfies the transcendental equation

$$(\pi r)^{1/2}\mathrm{e}^r\mathrm{erfc}(-r^{1/2}) = (1-2r)/2r \tag{36}$$

and has the value $r = 0.2040539\ldots$.

It is hoped that the method of derivation presented here may be of use in calculating STD's for other simple stochastic processes, and that the explicit solution given in equation (32) will find useful application in a range of problems. It is currently being employed in an ecological context, to calculate survival probabilities for organisms in the presence of a step-like drifting environmental boundary [14].

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Appendix

In this appendix we calculate the function $J_n(u)$, which satisfies the integro-difference equation

$$J_n(u) = \int_0^\infty \mathrm{d}u' \, \mathrm{e}^{-|u-u'|} J_{n-1}(u') \tag{A1}$$

with $J_0(u) = \mathrm{e}^{-u}$. This is accomplished by means of the generating function

$$\tilde{J}(u, z) = \sum_{n=0}^{\infty} z^n J_n(u). \tag{A2}$$

Summing over equation (A1) we find

$$\tilde{J}(u, z) = \mathrm{e}^{-u} + z \int_0^\infty \mathrm{d}u' \, \mathrm{e}^{-|u-u'|} \tilde{J}(u', z). \tag{A3}$$

Differentiating twice with respect to u yields the differential equation

$$\partial_u^2 \tilde{J}(u, z) = (1 - 2z)\tilde{J}(u, z). \tag{A4}$$

Demanding that $\tilde{J}(u, z)$ is finite as $u \to \infty$ gives

$$\tilde{J}(u, z) = \left(\frac{1 - (1 - 2z)^{1/2}}{z} \right) \exp[-(1 - 2z)^{1/2} u] \tag{A5}$$

where the z-dependent prefactor is found from substitution into equation (A3). The function $J_n(u)$ may be recovered from the generating function by means of the contour integral

$$J_n(u) = \oint_C \frac{\mathrm{d}z}{2\pi i} \frac{\tilde{J}(u, z)}{z^{n+1}} \tag{A6}$$

where the circular contour C runs counter-clockwise about the origin and excludes all singularities other than the pole at the origin.

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